What can we find from RREF?

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Outline

- RREF v.s. Linear Combination
- RREF v.s. Independent
- RREF v.s. Rank
- RREF v.s. Span

What can we find from RREF?

RREF v.s. Linear Combination

Column Correspondence Theorem

$$A = [\boldsymbol{a_1} \quad \cdots \quad \boldsymbol{a_n}] \xrightarrow{\text{RREF}} R = [\boldsymbol{r_1} \quad \cdots \quad \boldsymbol{r_n}]$$

If $\boldsymbol{a_j}$ is a linear combination of other columns of A ⇒ $\boldsymbol{r_j}$ is a linear combination of the ***corresponding*** columns of R with ***the same coefficients***

$\mathbf{a}_5 = -\mathbf{a}_1+\mathbf{a}_4$

$\mathbf{r}_5 = -\mathbf{r}_1+\mathbf{r}_4$

$\boldsymbol{a_j}$ is a linear combination of the ***corresponding*** columns of A with ***the same coefficients*** ⇐ If $\boldsymbol{r_j}$ is a linear combination of other columns of R

$\mathbf{a}_3 = 3\mathbf{a}_1\text{-}2\mathbf{a}_2$

$\mathbf{r}_3 = 3\mathbf{r}_1\text{-}2\mathbf{r}_2$

Column Correspondence Theorem - Example

$$A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \end{array} \qquad R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{a}_2 = 2\mathbf{a}_1 \quad \Longleftrightarrow \quad \mathbf{r}_2 = 2\mathbf{r}_1$$

$$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4 \quad \Longleftrightarrow \quad \mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$$

Column Correspondence Theorem – Intuitive Reason

$$a_1 + a_2 = a_3$$

$$A = \begin{bmatrix} 6 & 9 & 15 \\ 8 & 0 & 8 \\ 9 & 2 & 11 \end{bmatrix}$$

$$b_1 + b_2 = b_3$$

$$B = \begin{bmatrix} 9 & 2 & 11 \\ 8 & 0 & 8 \\ 6 & 9 & 15 \end{bmatrix}$$

$$c_1 + c_2 = c_3$$

$$C = \begin{bmatrix} 12 & 18 & 30 \\ 8 & 0 & 8 \\ 9 & 2 & 11 \end{bmatrix}$$

$$d_1 + d_2 = d_3$$

$$D = \begin{bmatrix} 6 & 9 & 15 \\ 8 & 0 & 8 \\ 3 & -7 & -4 \end{bmatrix}$$

Column Correspondence Theorem (Column 間的愛)：
就算 row elementary operation 讓 column 變的不同，
他們之間的關係永遠不變。

Column Correspondence Theorem – More Formal Reason

- Before we start:

Augmented Matrix: $\begin{bmatrix} A & b \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} R & b' \end{bmatrix}$

Coefficient Matrix: $A \xrightarrow{\text{RREF}} R$

$$\underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array}\right]}_{\begin{bmatrix} A & b \end{bmatrix}} \Rightarrow \underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\begin{bmatrix} R & b' \end{bmatrix}}$$

Column Correspondence Theorem – More Formal Reason

- The RREF of matrix A is R

  $Ax = b$ and $Rx = b$ have the same solution set?

- The RREF of augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ is $\begin{bmatrix} R & b' \end{bmatrix}$

  $Ax = b$ and $Rx = b'$ have the same solution set

- The RREF of matrix A is R

  $Ax = 0$ and $Rx = 0$ have the same solution set

If $b = 0$, then $b' = 0$.

Column Correspondence Theorem

- The RREF of matrix A is R, $Ax = 0$ and $Rx = 0$ have the same solution set

$$
A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \end{array}
\qquad
R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
$$

$\mathbf{a}_2 = 2\mathbf{a}_1$

$-2\mathbf{a}_1+\mathbf{a}_2=\mathbf{0}$

$\boxed{Ax = 0}$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{Rx = 0}$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{r}_2 = 2\mathbf{r}_1$

$-2\mathbf{r}_1+\mathbf{r}_2=\mathbf{0}$

Column Correspondence Theorem

- The RREF of matrix A is R, $Ax = 0$ and $Rx = 0$ have the same solution set

$$A = \begin{array}{c} \begin{array}{cccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \end{array} \qquad R = \begin{array}{c} \begin{array}{cccccc} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{r}_4 & \mathbf{r}_5 & \mathbf{r}_6 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4$ $\Updownarrow$ $\mathbf{a}_1 - \mathbf{a}_4 + \mathbf{a}_5 = 0$ $\Leftrightarrow$ $\boxed{Ax = 0}$ $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ $\Leftrightarrow$ $\boxed{Rx = 0}$ $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ $\Leftrightarrow$ $\mathbf{r}_1 - \mathbf{r}_4 + \mathbf{r}_5 = 0$ $\Updownarrow$ $\mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

How about Rows?

- Are there row correspondence theorem? NO

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \text{—} \; \boldsymbol{a}_1^T \; \text{—} \\ \text{—} \; \boldsymbol{a}_2^T \; \text{—} \\ \text{—} \; \boldsymbol{a}_3^T \; \text{—} \\ \text{—} \; \boldsymbol{a}_4^T \; \text{—} \end{bmatrix} \qquad R = \begin{bmatrix} \text{—} \; \boldsymbol{r}_1^T \; \text{—} \\ \text{—} \; \boldsymbol{r}_2^T \; \text{—} \\ \text{—} \; \boldsymbol{r}_3^T \; \text{—} \\ \text{—} \; \boldsymbol{r}_4^T \; \text{—} \end{bmatrix}$$

$$Span\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\} = Span\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4\}$$

Are they the same?

Span of Columns

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = [\boldsymbol{a_1} \quad \cdots \quad \boldsymbol{a_6}] \qquad R = [\boldsymbol{r_1} \quad \cdots \quad \boldsymbol{r_6}]$$

$$Span\,\{\boldsymbol{a_1}, \quad \cdots \quad , \boldsymbol{a_6}\} \qquad Span\,\{\boldsymbol{r_1}, \quad \cdots \quad , \boldsymbol{r_6}\}$$

Are they the same?

The elementary row operations change the span of columns.

NOTE

- Original Matrix v.s. RREF
  - Columns:
    - The relations between the columns are the same.
    - The span of the columns are different.
  - Rows:
    - The relations between the rows are changed.
    - The span of the rows are the same.

What can we find from RREF?

RREF v.s. Independent

Column Correspondence Theorem

pivot columns

Leading entries

$$A = \begin{bmatrix} \boxed{\begin{matrix}1\\-1\\2\\-3\end{matrix}} & \begin{matrix}2\\-2\\4\\-6\end{matrix} & \boxed{\begin{matrix}-1\\1\\-3\\2\end{matrix}} & \boxed{\begin{matrix}2\\2\\2\\0\end{matrix}} & \begin{matrix}1\\3\\0\\3\end{matrix} & \begin{matrix}2\\6\\3\\9\end{matrix} \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

linear independent

linear independent

The pivot columns are linear independent.

Column Correspondence Theorem

You can prove unique RREF by these properties

pivot columns

Leading entries

$$A = \begin{bmatrix} \boxed{\begin{matrix} 1 \\ -1 \\ 2 \\ -3 \end{matrix}} & \begin{matrix} 2 \\ -2 \\ 4 \\ -6 \end{matrix} & \boxed{\begin{matrix} -1 \\ 1 \\ -3 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 2 \\ 2 \\ 0 \end{matrix}} & \begin{matrix} 1 \\ 3 \\ 0 \\ 3 \end{matrix} & \begin{matrix} 2 \\ 6 \\ 3 \\ 9 \end{matrix} \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{a}_2 = 2\mathbf{a}_1$

$\mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_4$

$\mathbf{a}_6 = -5\mathbf{a}_1 - 3\mathbf{a}_3 + 2\mathbf{a}_4$

$\mathbf{r}_2 = 2\mathbf{r}_1$

$\mathbf{r}_5 = -\mathbf{r}_1 + \mathbf{r}_4$

$\mathbf{r}_6 = -5\mathbf{r}_1 - 3\mathbf{r}_3 + 2\mathbf{r}_4$

The non-pivot columns are the linear combination of the previous pivot columns.

Column Correspondence Theorem

Given the pivot columns of a matrix and its RREF, we can reconstruct the whole matrix.

pivot columns

$$A = \begin{array}{c} \left[\begin{array}{cccccc} \boxed{\begin{matrix}1\\-1\\2\\-3\end{matrix}} & & \boxed{\begin{matrix}-1\\1\\-3\\2\end{matrix}} & \boxed{\begin{matrix}2\\2\\2\\0\end{matrix}} & & \end{array}\right] \\ \begin{array}{cccccc} \mathbf{a}_1 & & \mathbf{a}_3 & \mathbf{a}_4 & & \end{array} \end{array} \qquad R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{a}_2 = 2\mathbf{a}_1$ $\qquad$ $\mathbf{r}_2 = 2\mathbf{r}_1$

$\mathbf{a}_5 = -\mathbf{a}_1+\mathbf{a}_4$ $\qquad$ $\mathbf{r}_5 = -\mathbf{r}_1+\mathbf{r}_4$

$\mathbf{a}_6 = -5\mathbf{a}_1-3\mathbf{a}_3+2\mathbf{a}_4$ $\qquad$ $\mathbf{r}_6 = -5\mathbf{r}_1-3\mathbf{r}_3+2\mathbf{r}_4$

Independent

The columns are independent

$\Downarrow$

Every column is a pivot columns

$\Downarrow$

Every column in RREF(A) is standard vector.

3X3

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Columns are linear independent

$\Big\Downarrow$ RREF

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Identity matrix

Independent

The columns are independent

Every column is a pivot columns

Every column in RREF(A) is standard vector.

4X3

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

Columns are linear independent

RREF

$$\begin{bmatrix} \\ \\ \end{bmatrix} \quad \begin{bmatrix} I \\ O \end{bmatrix}$$

Independent

The columns are independent

Every column is a pivot columns

Every column in RREF(A) is standard vector.

3X4

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Columns are linea
independent

RREF

Cannot be a pivot column

Independent

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{matrix} \Longrightarrow \\ \not\Longleftarrow \end{matrix} \text{The columns are dependent}$$

(矮胖型)

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Dependent or Independent?

More than 3 vectors in R^3 must be dependent.

More than m vectors in R^m must be dependent.

What can we find from RRef?

RREF v.s. Rank

Rank

Pivot column → Leading Entry → Non-zero row

Maximum number of Independent Columns

||

Number of Pivot Column

||

Number of Non-zero rows

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

Rank = ? **3**

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = ? **3**

Properties of Rank from RREF

Maximum number of Independent Columns $\Rightarrow$ Rank A $\leq$ Number of columns

$=$

Number of Pivot Column

$=$

Number of Non-zero rows $\Rightarrow$ Rank A $\leq$ Number of rows

$\Downarrow$

Rank A $\leq$ Min(Number of columns, Number of rows)

Properties of Rank from RREF

Matrix A is ***full rank*** if Rank A = min(m,n)

- Given a mxn matrix A:
  - Rank A $\leq$ min(m, n)
  - Because "the columns of A are independent" is equivalent to "rank A = n"
    - If m < n, the columns of A is dependent.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

3 X 4

Rank A $\leq$ 3

$$\left\{ \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \begin{bmatrix} * \\ * \\ * \end{bmatrix} \right\}$$

A matrix set has 4 vectors belonging to R^3 is dependent

- In R^m, you cannot find more than m vectors that are independent.

Basic, Free Variables v.s. Rank

$$Ax = b$$

$$\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + 5x_5 &= 2 \\ -x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\ 2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\ -3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 2 & 5 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{array}\right]$$

A b

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF(A) b'

$$\begin{aligned} x_1 + 2x_2 - x_5 &= -5 \\ x_3 &= -3 \\ x_4 + x_5 &= 2 \\ \cancel{0} &\cancel{= 0} \end{aligned}$$

3 useful equations

$$x_1 = -5 - 2x_2 + x_5$$

$$x_3 = -3$$

$$x_4 = 2 - x_5$$

rank = non-zero row = 3 basic variables

nullity = No. column – non-zero row = 2 free variables

Rank

Rank

Maximum number of Independent Columns

Number of Pivot Column

Number of Non-zero rows

Number of Basic Variables

Nullity = no. column - rank

~~Number ero rows~~

Number of Free Equations

What can we find from RREF?

RREF v.s. Span

Consistent or not

- Given Ax=b, if the reduced row echelon form of [A b] is

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Consistent

b is in the span of the columns of A

- Given Ax=b, if the reduced row echelon form of [A b] is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

inconsistent

b is NOT in the span of the columns of A

Consistent or not

Ax =b is inconsistent (no solution)

The RREF of [A b] is

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ \boxed{0 & 0 & 0 & 0 & 0 & d} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad d \neq 0$$

Only the last column is non-zero

Rank A $\neq$ rank [A b]

Need to know b

Consistent or not

Ax =b is consistent for ***every*** b

$\Updownarrow$

RREF of [A b] cannot have a row whose only non-zero entry is at the last column

$\Updownarrow$

RREF of A cannot have zero row

$\Updownarrow$

Rank A = no. of rows

Consistent or not

e.g. $\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$

3 independent columns

Ax =b is consistent for ***every*** b $\Leftrightarrow$ Rank A = no. of rows

Every b is in the span of the columns of A= $[a_1 \quad \cdots \quad a_n]$

Every b belongs to $Span\{a_1, \quad \cdots \quad , a_n\}$

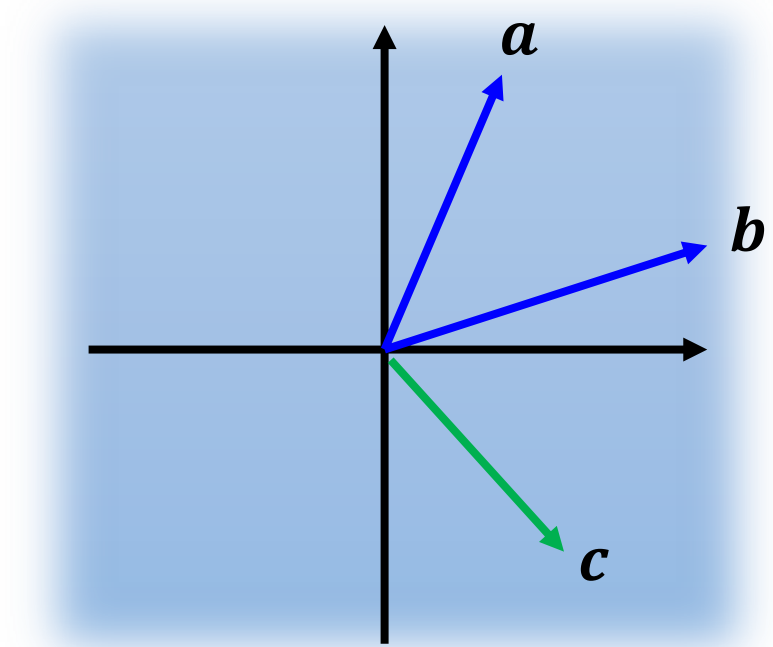
$Span\{a_1, \quad \cdots \quad , a_n\} = R^m$

m independent vectors can span R^m

$\Leftrightarrow$ More than m vectors in R^m must be dependent.

m independent vectors can span R^m	⟺	More than m vectors in R^m must be dependent.

- Consider R^2



$$\text{Does } \mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ generate } \mathcal{R}^3?$$

yes

independent

Full Rank: Rank = n & Rank = m

- The size of A is mxn

Rank A = n

A is square or 高瘦

$$\begin{bmatrix} \boxed{1} & \boxed{0} & * & * & * & * \\ \boxed{0} & \boxed{1} & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{0} & \boxed{0} & * & * & * & * \\ \boxed{0} & \boxed{0} & * & * & * & * \end{bmatrix}$$

The columns of A are linearly independent.

$A\mathbf{x} = \mathbf{b}$ has at most one solution

All columns are pivot columns.

RREF of A: $\begin{bmatrix} I \\ O \end{bmatrix}$

Full Rank: Rank = n & Rank = m

- The size of A is mxn

Rank A = m

A is square or 矮胖

$$\begin{bmatrix} \boxed{1} & * & * & * & * & * \\ \boxed{0} & \boxed{1} & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & * & * \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & * \end{bmatrix}$$

Every row of *R* contains a pivot position (leading entry).

$A\mathbf{x} = \mathbf{b}$ always have solution (at least one solution) for every $\mathbf{b}$ in $\mathcal{R}^m$.

The columns of *A* generate $\mathcal{R}^m$.

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- 感謝 劉俊廷 同學發現投影片 上的錯誤